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On the construction of particular solutions to (1 + 1)-dimensional partial differential equations

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Abstract
This paper presents a modification of the dressing method which allows one to construct the particular solutions for certain classes of (1 + 1)-dimensional systems of partial differential equations, which are nonintegrable in the general case. Examples of one- and two-soliton equations are given.

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1. Introduction

Completely integrable (1 + 1)-dimensional nonlinear partial differential equations (PDEs) have a wide application in mathematical physics. For instance, the nonlinear Shrödinger equation describes propagation of a signal in a nonlinear fibre [1, 2], and the Korteweg–de Vries (KdV) equation [3–6] and the Camassa–Holm (CH) equation [7–10] represent two different long-wave approximations of the shallow-water system. There are many methods of investigation applicable for (1 + 1)-dimensional integrable equations. In this paper we deal with so-called dressing methods [11–17] which have been developed for the construction of particular solutions of completely integrable nonlinear PDEs.

Although completely integrable nonlinear equations reflect many features of physical systems, they represent only certain approximations of the original equations of mathematical physics, which are nonintegrable in general. The methods of analytical study of nonintegrable nonlinear PDEs are not well developed yet. In this paper we consider the problem of the construction of the particular solutions for (1 + 1)-dimensional systems of nonlinear PDEs without relation to the complete integrability. The method discussed below is a development of the dressing method based on the algebraic system of equations [18, 19]

$$L(\psi_m) \equiv \psi_m - \sum_{n=1}^{N} \psi_n R_{nm} = \eta_m \qquad m = 1, \ldots, N \tag{1}$$

or matrix equation

$$\begin{aligned} &L(\psi) \equiv \psi(\boldsymbol{I} - \boldsymbol{R}) = \eta \qquad \det(\boldsymbol{I} - \boldsymbol{R}) \neq 0, \\ &\psi = [\psi_1 \ldots \psi_N] \qquad \eta = [\eta_1 \ldots \eta_N] \end{aligned} \tag{2}$$

which in turn is a discrete version of the classical dressing method based on the $\bar{\partial}$-problem [13–15]. Here $\boldsymbol{I}$ is an $N \times N$ identity matrix and $\boldsymbol{R}$ is an $N \times N$ matrix whose properties will be discussed below; η is the $1 \times N$ row which represents the right-hand side of the matrix equation. The structure of the matrices $\boldsymbol{R}$ and η defines the particular system of nonlinear PDEs, related with the given matrix equation (2).

The algorithm represented in [18, 19] relates the matrix equation (2) with the $(n + 1)$-dimensional system of nonlinear PDEs, where $n > 1$. Some additional requirements on the matrix $\boldsymbol{R}$ allow one to use the same algorithm for the study of (1 + 1)-dimensional systems of nonlinear PDEs. In section 2 we present this algorithm for a particular choice of the matrices $\boldsymbol{R}$ and η.

It is known that completely integrable equations admit n-parametric solutions, such as n-soliton and n-kink (n is any integer). We will show (section 3) that even the requirement for the nonlinear PDEs to possess one-parametric solution (either soliton or kink) generates many restrictions on the coefficients of the general equation derived in section 2. Examples of scalar equations which admit one- and two-soliton solutions are presented in section 3. Conclusions are given in section 4. To avoid cumbersome formulae in the main text, we collect important intermediate results in an appendix.

2. Dressing method for (1 + 1)-dimensional nonlinear PDEs

We present the algorithm which relates the matrix equation (2) with a particular system of nonlinear PDEs. The form of the nonlinear system is defined by the matrices $\boldsymbol{R}$ and η of the algebraic system.

First of all, one needs to introduce parameters x and t (independent variables of the nonlinear system) into equation (2). This can be done in a number of ways. We consider an example which results in the generalization of the KdV equation and the Burgers equation. For this purpose let us introduce parameters x and t by the following formulae:

$$\boldsymbol{R}_x = \boldsymbol{f}\boldsymbol{g} \tag{3}$$

$$\begin{aligned} &\boldsymbol{R}_t = \boldsymbol{f}_0\boldsymbol{g} + \boldsymbol{f}_1\boldsymbol{g}_x + \boldsymbol{f}_2\boldsymbol{g}_{xx} \qquad \eta = \boldsymbol{g} \\ &\boldsymbol{f} = [f_1, \ldots, f_N]^T \qquad \boldsymbol{g} = [g_1, \ldots, g_N] \\ &\boldsymbol{f}_0 = \gamma_{00}\boldsymbol{f} + \gamma_{01}\boldsymbol{f}_x + \gamma_{02}\boldsymbol{f}_{xx} \qquad \boldsymbol{f}_1 = \gamma_{10}\boldsymbol{f} + \gamma_{11}\boldsymbol{f}_x \qquad \boldsymbol{f}_2 = \gamma_{20}\boldsymbol{f} \\ &\gamma_{ij} = \text{const} \end{aligned} \tag{4}$$

with

$$\begin{aligned} &\boldsymbol{f}_t = \kappa_{10}\boldsymbol{f} + \kappa_{11}\boldsymbol{f}_x + \kappa_{12}\boldsymbol{f}_{xx} + \kappa_{13}\boldsymbol{f}_{xxx} \\ &\boldsymbol{g}_t = \kappa_{20}\boldsymbol{g} + \kappa_{21}\boldsymbol{g}_x + \kappa_{22}\boldsymbol{g}_{xx} + \kappa_{23}\boldsymbol{g}_{xxx} \qquad \kappa_{ij} = \text{const.} \end{aligned} \tag{5}$$

Equations (5) are linear differential equations for the functions $\boldsymbol{g}$ and $\boldsymbol{f}$. They are resolvable for all values of parameters κ_{ij}. The system (3), (4) is overdetermined with compatibility condition given by the formula

$$\partial_t(\boldsymbol{f}\boldsymbol{g}) = \partial_x(\boldsymbol{f}_0\boldsymbol{g} + \boldsymbol{f}_1\boldsymbol{g}_x + \boldsymbol{f}_2\boldsymbol{g}_{xx}). \tag{6}$$

This equation produces relations among parameters γ_{ij} and κ_{ij} and restrictions on the solutions of equations (5) (in general).

Equations (3)–(5), which introduce the dependence on the parameters x and t into the matrix equation (2), are fundamental. These equations together with compatibility condition (6) represent the *first factor* which specifies the nonlinear system.

Now one needs to construct the auxiliary linear differential equation [18] (similar to the classical dressing method [13–15]). For this purpose we take it that the following system of linear matrix equations is held together with the original equation (2):

$$L(\psi_x) = U_0 \boldsymbol{g} + \boldsymbol{g}_x \tag{7}$$

$$L(\psi_{xx}) = (2U_{0x} - U_1)\boldsymbol{g} + U_0 \boldsymbol{g}_x + \boldsymbol{g}_{xx} \tag{8}$$

$$L(\psi_{xxx}) = (3U_{0xx} - 3U_{1x} + U_2)\boldsymbol{g} + (3U_{0x} - U_1)\boldsymbol{g}_x + U_0 \boldsymbol{g}_{xx} + \boldsymbol{g}_{xxx} \tag{9}$$

$$\begin{aligned} &L(\psi_t) = (U_0^{(0)} + \kappa_{20})\boldsymbol{g} + (U_0^{(1)} + \kappa_{21})\boldsymbol{g}_x + (U_0^{(2)} + \kappa_{22})\boldsymbol{g}_{xx} + \kappa_{23}\boldsymbol{g}_{xxx} \\ &U_0^{(0)} = \psi \boldsymbol{f}_0 = \gamma_{00} U_0 + \gamma_{01} U_1 + \gamma_{02} U_2 \\ &U_0^{(1)} = \psi \boldsymbol{f}_1 = \gamma_{10} U_0 + \gamma_{11} U_1 \qquad U_0^{(2)} = \psi \boldsymbol{f}_2 = \gamma_{20} U_0 \end{aligned} \tag{10}$$

$$U_n = \psi \partial_x^n \boldsymbol{f} \qquad n = 1, 2, \ldots. \tag{11}$$

For instance, equation (7) follows from equation (2) if one differentiates it with respect to x and uses equation (3) for $\boldsymbol{R}_x$. To derive equation (8) one needs to differentiate equation (7) with respect to x and use equation (3) for $\boldsymbol{R}_x$, and so on. Having the system (2), (7)–(10) and using the superposition principle for the linear matrix equations one can construct the *homogeneous* matrix equation

$$L(\psi_t - \kappa_{23}\psi_{xxx} + W_2\psi_{xx} + W_1\psi_x + W_0\psi) = 0 \tag{12}$$

where W_k are expressed through the functions U_n by the formulae

$$\begin{aligned} &W_2 = -\kappa_{22} - U_0^{(2)} + \kappa_{23} U_0 \\ &W_1 = -\kappa_{21} - U_0^{(1)} + \kappa_{23}(3U_{0x} - U_1) - W_2 U_0 \\ &W_0 = -\kappa_{20} - U_0^{(0)} + \kappa_{23}(3U_{0xx} - 3U_{1x} + U_2) - W_2(2U_{0x} - U_1) - W_1 U_0. \end{aligned} \tag{13}$$

Since $\det(\boldsymbol{I} - \boldsymbol{R}) \neq 0$ in equation (2), the only solution of the matrix homogeneous equation $L(\chi) = 0$ is $\chi \equiv 0$. Then equation (12) produces the following linear PDE for the function ψ:

$$\psi_t - \kappa_{23}\psi_{xxx} + W_2\psi_{xx} + W_1\psi_x + W_0\psi = 0. \tag{14}$$

The nonlinear system for the functions U_n (see equation (11)) can be derived from the linear equation (14) through multiplication by the vector $\partial_x^n \boldsymbol{f}$:

$$\begin{aligned} U_{nt} &- \kappa_{10}U_n - \kappa_{11}U_{n+1} - \kappa_{12}U_{n+2} - \kappa_{13}U_{n+3} - \kappa_{23}(U_{nxxx} - 3U_{n+1xx} + 3U_{n+2x} - U_{n+3}) \\ &+ W_2(U_{nxx} - 2U_{n+1x} + U_{n+2}) + W_1(U_{nx} - U_{n+1}) + W_0 U_n = 0 \\ &n = 0, 1, 2, \ldots. \end{aligned} \tag{15}$$

Equation (15) is a differential–difference equation with continuous variables x, t and discrete variable n. To reduce it to the system of pure PDEs, one needs to consider U_n $(n = 0, 1, \ldots)$ as the set of different functions of variables x and t with subscript n and establish an additional relation among the functions $U_n(x, t)$ with different n. For this purpose we construct another differential-difference equation on the function U_n, $n = 1, 2, \ldots$ (compare [18, 19]). The straightforward method is to require that the matrix $\boldsymbol{R}$ satisfies the matrix equation of the following general form:

$$\boldsymbol{R}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{R} + \sum_k \boldsymbol{f}_k \boldsymbol{g}_k \qquad \boldsymbol{f}_k = [f_1^{(k)}, \ldots, f_N^{(k)}]^T \qquad \boldsymbol{g}_k = [g_1^{(k)}, \ldots, g_N^{(k)}] \tag{16}$$

$$\boldsymbol{A}\boldsymbol{f} = M_f^{(k)}\boldsymbol{f} \qquad \boldsymbol{g}\boldsymbol{A} = M_g^{(k)}\boldsymbol{g} \tag{17}$$

where $\boldsymbol{A} = \{a_{ij}\}$ is a constant $N \times N$ matrix which will be specified below; $\boldsymbol{f}_k = L_f^{(k)}\boldsymbol{f}$, $\boldsymbol{g}_k = L_g^{(k)}\boldsymbol{g}$; operators $L_f^{(k)}$, $L_g^{(k)}$, $M_f^{(k)}$, $M_g^{(k)}$ are some linear differential operators. To achieve the generalization of the KdV and Burgers equations one needs to take equations (16), (17) in the following form:

$$\boldsymbol{R}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{R} + \boldsymbol{f}_3\boldsymbol{g} + \boldsymbol{f}_4\boldsymbol{g}_x \qquad \boldsymbol{f}_3 = \gamma_{30}\boldsymbol{f} + \gamma_{31}\boldsymbol{f}_x \qquad \boldsymbol{f}_4 = \gamma_{40}\boldsymbol{f} + \gamma_{41}\boldsymbol{f}_x \tag{18}$$

$$\begin{aligned} &\boldsymbol{A}\boldsymbol{f} = \kappa_{30}\boldsymbol{f} + \kappa_{31}\boldsymbol{f}_x + \kappa_{32}\boldsymbol{f}_{xx} \qquad \boldsymbol{g}\boldsymbol{A} = \kappa_{40}\boldsymbol{g} + \kappa_{41}\boldsymbol{g}_x + \kappa_{42}\boldsymbol{g}_{xx} \\ &\kappa_{ij} = \text{const.} \qquad \gamma_{ij} = \text{const.} \end{aligned} \tag{19}$$

Equations (19) fix the x-dependence of functions $\boldsymbol{f}$. Equation (18) should be compatible with equations (3), (4), which is provided by the additional condition

$$\boldsymbol{f}\boldsymbol{g}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{f}\boldsymbol{g} = \partial_x(\boldsymbol{f}_3\boldsymbol{g} + \boldsymbol{f}_4\boldsymbol{g}_x) \tag{20}$$

which gives rise to the relations between elements of the matrix $\boldsymbol{A}$ and parameters κ_{ij}, γ_{ij}. Equations (18) and (19) produce the second differential-difference equation on the functions U_n.

We start construction of the second differential-difference equation with the construction of the second auxiliary linear problem. This is possible because equation (18) provides one more matrix equation:

$$L(\psi\boldsymbol{A}) = (U_0^{(3)} + \kappa_{40})\boldsymbol{g} + (U_0^{(4)} + \kappa_{41})\boldsymbol{g}_x + \kappa_{42}\boldsymbol{g}_{xx} \tag{21}$$

which follows from equation (2) if one multiplies it by the matrix $\boldsymbol{A}$ and uses formulae (18), (19). By using the superposition of the linear matrix equations (2), (7), (8), (21) one can construct the following homogeneous matrix equation:

$$L(\psi A - \kappa_{42}\psi_{xx} + V_1\psi_x + V_0\psi) = 0 \tag{22}$$

where V_k are given by the formulae

$$\begin{aligned} &V_1 = -\kappa_{41} - U_0^{(4)} + \kappa_{42}U_0 \\ &V_0 = -\kappa_{40} - U_0^{(3)} + \kappa_{42}(2U_{0x} - U_1) - V_1U_0 \\ &U_0^{(3)} = \gamma_{30}U_0 + \gamma_{31}U_1 \qquad U_0^{(4)} = \gamma_{40}U_0 + \gamma_{41}U_1. \end{aligned} \tag{23}$$

The consequence of this equation is another linear PDE on the function ψ:

$$\psi A - \kappa_{42}\psi_{xx} + V_1\psi_x + V_0\psi = 0. \tag{24}$$

The nonlinear differential-difference equation can be derived by multiplication of equation (24) by the vector $\partial_x^n\boldsymbol{f}$ and using the definition of the functions U_n given by equation (11):

$$(\kappa_{32} - \kappa_{42})U_{n+2} + \kappa_{31}U_{n+1} + \kappa_{30}U_n - \kappa_{42}(U_{nxx} - 2U_{n+1x}) + V_1(U_{nx} - U_{n+1}) + V_0U_n = 0. \tag{25}$$

Now the complete system of nonlinear PDEs on the functions U_n, $(n = 1, \ldots, 5)$ is represented by equations (15) with $n = 0, 1$ and (25) with $n = 0, 1, 2, 3$.

If, in addition,

$$\kappa_{42} = \kappa_{32}, \tag{26}$$

then the complete system of nonlinear PDEs on the functions $u = U_0$, $u_1 = U_1$, $u_2 = U_2$ is represented by the system (15) with $n = 0$ and (25) with $n = 0, 1, 2$:

$$\begin{aligned} &u_t - \kappa_{10}u - \kappa_{11}u_1 - \kappa_{12}u_2 - \kappa_{13}u_3 - \kappa_{23}(u_{xxx} - 3u_{1xx} + 3u_{2x} - u_3) \\ &\qquad + W_2(u_{xx} - 2u_{1x} + u_2) + W_1(u_x - u_1) + W_0u = 0 \end{aligned} \tag{27}$$

$$\kappa_{31}u_1 + \kappa_{30}u - \kappa_{42}(u_{xx} - 2u_{1x}) + V_1(u_x - u_1) + V_0u = 0 \tag{28}$$

$$\kappa_{31}u_2 + \kappa_{30}u_1 - \kappa_{42}(u_{1xx} - 2u_{2x}) + V_1(u_{1x} - u_2) + V_0u_1 = 0 \tag{29}$$

$$\kappa_{31}u_3 + \kappa_{30}u_2 - \kappa_{42}(u_{2xx} - 2u_{3x}) + V_1(u_{2x} - u_3) + V_0u_2 = 0 \tag{30}$$

where W_i and V_i are given by equations (13) and (23).

2.1. Reduction to the single nonlinear equation

The system of equations (27)–(30) admits reduction which results in the single PDE. In fact, one can check that equation (28) admits solution in the form

$$u_1 = \alpha_1 u + \alpha_2 u_x + \alpha_3 u^2 \tag{31}$$

provided that α_i, $i = 1, 2, 3$, satisfy the following system of equations:

$$(\alpha_2 - 1)\alpha_2\gamma_{41} = 0 \tag{32}$$
$$(\alpha_2(1 + 2\alpha_3) - \alpha_3)\gamma_{41} = 0 \tag{33}$$
$$(\alpha_3 + 1)\alpha_3\gamma_{41} = 0 \tag{34}$$
$$(2\alpha_2 - 1)\kappa_{32} = 0 \tag{35}$$
$$\alpha_3(\gamma_{40} - \gamma_{31} + 2\alpha_1\gamma_{41} - 2\kappa_{32}) + \gamma_{40} + \alpha_1\gamma_{41} - \kappa_{32} = 0 \tag{36}$$
$$\alpha_2(\gamma_{40} - \gamma_{31} + 2\alpha_1\gamma_{41} - 2\kappa_{32}) - \gamma_{40} - \alpha_1\gamma_{41} + (3 + 4\alpha_3)\kappa_{32} = 0 \tag{37}$$
$$\kappa_{30} - \kappa_{40} + \alpha_1(\kappa_{31} + \kappa_{41}) = 0 \tag{38}$$
$$\alpha_2(\kappa_{31} + \kappa_{41}) + 2\alpha_1\kappa_{32} - \kappa_{41} = 0 \tag{39}$$
$$\alpha_1(\gamma_{40} - \gamma_{31} - 2\kappa_{32}) - \gamma_{30} + \alpha_1^2\gamma_{41} + \alpha_3\kappa_{31} + \kappa_{41} + \alpha_3\kappa_{41} = 0. \tag{40}$$

If one substitutes u_1, given by equation (31), and u_{2x} from equation (29) into (27), and imposes the following relations:

$$\kappa_{23} - \kappa_{13} = 0 \qquad 3\alpha_2\gamma_{41}\kappa_{23} = 0 \qquad 3\alpha_3\gamma_{41}\kappa_{23} = 0 \tag{41}$$
$$\kappa_{32}(\kappa_{23} - 2\gamma_{02} - 2\gamma_{20}) + 3\gamma_{40}\kappa_{23} + 3\alpha_1\gamma_{41}\kappa_{23} = 0 \tag{42}$$
$$3\kappa_{23}(\kappa_{31} + \kappa_{41}) - 2\kappa_{32}(\kappa_{12} + \kappa_{22}) = 0 \tag{43}$$
$$\kappa_{32} \neq 0 \tag{44}$$

in addition to the relations (32)–(40), then all terms with u_2 and u_3 disappear from equation (27), resulting in the following equation on the function u:

$$u_t + r_{11}u_x + r_1u + r_2u_{xx} + r_3u_{xxx} + r_4u_x^2 + r_5uu_x + r_6uu_{xx} + r_7u^2 + r_8u^2u_x + r_9u^3 + r_{10}u^4 = 0 \tag{45}$$

where arbitrary parameters r_i are expressed in terms of parameters κ_{ij}, γ_{ij} and α_i by the formulae (97)–(107). Hereafter we assume that $r_{11} = 0$, since the corresponding term in equation (45) can be eliminated by redefinition of the variable t.

Note that equation (45) has been derived only by using equations (2)–(5) and (18), (19). We have not used the compatibility conditions (6) and (20), which are necessary in order to provide the non-empty manifold of particular solutions for equation (45). It will be shown in the next section that the compatibility conditions give rise to the relations among parameters r_i.

3. Particular solutions and related reductions of equation (45)

Important reductions of the nonlinear equation (45) are produced by the compatibility conditions (6) and (20) whose exact forms depend on both the structure of matrix $\boldsymbol{A}$ and the dimension N of the matrix equation (2). Hereafter we will deal with the diagonal matrix $\boldsymbol{A} = \mathrm{diag}(a_1, \ldots, a_N)$.

Note that equations (5), (19) admit the functions $\boldsymbol{f}$ and $\boldsymbol{g}$ in the following form:

$$\boldsymbol{f} = \begin{bmatrix} f_1\mathrm{e}^{k_{11}x + k_{12}t} \\ \vdots \\ f_N\mathrm{e}^{k_{N1}x + k_{N2}t} \end{bmatrix} \qquad \boldsymbol{g} = [\, g_1\mathrm{e}^{\omega_{11}x + \omega_{12}t} \quad \ldots \quad g_N\mathrm{e}^{\omega_{N1}x + \omega_{N2}t} \,] \tag{46}$$

where ω_{ij} and k_{ij} are solutions of the characteristic equations, related with the differential equations (5), (19):

$$a_i = \kappa_{30} + \kappa_{31}k_{i1} + \kappa_{32}k_{i1}^2 \tag{47}$$

$$a_i = \kappa_{40} + \kappa_{41}\omega_{i1} + \kappa_{42}\omega_{i1}^2 \tag{48}$$

$$\omega_{i2} = \kappa_{20} + \omega_{i1}\kappa_{21} + \omega_{i1}^2\kappa_{22} + \omega_{i1}^3\kappa_{23} \tag{49}$$

$$k_{i2} = \kappa_{10} + k_{i1}\kappa_{11} + k_{i1}^2\kappa_{12} + \omega_{i1}^3\kappa_{13} \qquad i = 1, \ldots, N. \tag{50}$$

Let

$$\kappa_{30} = (\kappa_{31}^2 + 4\kappa_{32}\kappa_{40} - \kappa_{41}^2)/(4\kappa_{32}) \tag{51}$$

to simplify solution of the above characteristic equations, and introduce parameters b_i instead of a_i by the formula

$$a_i = (b_i^2\kappa_{42}^2 + 4\kappa_{42}\kappa_{40} - \kappa_{41}^2)/(4\kappa_{42}) \qquad i = 1, \ldots, N. \tag{52}$$

Then the compatibility conditions (6) and (20) can be represented in the following form:

$$\begin{aligned}
&(\omega_{i1} + k_{i1})(\gamma_{30} + \gamma_{40}\omega_{i1} + \gamma_{31}k_{i1} + \gamma_{41}\omega_{i1}k_{i1}) = 0\\
&\frac{\kappa_{32}}{4}(b_i^2 - b_j^2) - \gamma_{30}\omega_{i1} - \gamma_{40}\omega_{i1}^2 - \gamma_{30}k_{j1} - \gamma_{31}\omega_{i1}k_{j1}\\
&\qquad -\gamma_{40}\omega_{i1}k_{j1} - \gamma_{41}\omega_{i1}^2k_{j1} - \gamma_{31}k_{j1}^2 - \gamma_{41}\omega_{i1}k_{j1}^2 = 0\\
&\gamma_{00}\omega_{i1} + \gamma_{10}\omega_{i1}^2 + \gamma_{20}\omega_{i1}^3 - \omega_{i2} + \gamma_{00}k_{i1} + \gamma_{01}\omega_{i1}k_{i1} + \gamma_{10}\omega_{i1}k_{i1} + \gamma_{11}\omega_{i1}^2k_{i1}\\
&\qquad +\gamma_{20}\omega_{i1}^2k_{i1} + \gamma_{01}k_{i1}^2 + \gamma_{02}\omega_{i1}k_{i1}^2 + \gamma_{11}\omega_{i1}k_{i1}^2 + \gamma_{02}k_{i1}^3 - k_{i2} = 0\\
&\gamma_{00}\omega_{i1} + \gamma_{10}\omega_{i1}^2 + \gamma_{20}\omega_{i1}^3 - \omega_{i2} + \gamma_{00}k_{j1} + \gamma_{01}\omega_{i1}k_{j1} + \gamma_{10}\omega_{i1}k_{j1} + \gamma_{11}\omega_{i1}^2k_{j1}\\
&\qquad +\gamma_{20}\omega_{i1}^2k_{j1} + \gamma_{01}k_{j1}^2 + \gamma_{02}\omega_{i1}k_{j1}^2 + \gamma_{11}\omega_{i1}k_{j1}^2 + \gamma_{02}k_{j1}^3 - k_{j2} = 0\\
&\qquad i, j = 1, \ldots, N \quad i \neq j
\end{aligned} \tag{53}$$

which will be used hereafter. One can see that the number of equations in system (53) increases with dimension N of the matrix equation (2) and equals $2N^2$. Depending on the number of equations in system (53), one has different relations among parameters κ_{ij} and γ_{ij} and, consequently, different reductions for the nonlinear PDE (45). Two examples with $N = 1$ and 2 will be considered below.

Note that along with formula (11) the following relation may be used for the construction of particular solutions of equation (45)

$$u = -\partial_x \ln(\det(\boldsymbol{I} - \boldsymbol{R})) \tag{54}$$

which is more convenient for calculations. Here $\boldsymbol{I}$ is an $N \times N$ identity matrix and matrix $\boldsymbol{R}$ is related with $\boldsymbol{f}$ and $\boldsymbol{g}$ by the equation

$$\boldsymbol{R} = \partial_x^{-1}\boldsymbol{f}\boldsymbol{g} \tag{55}$$

which follows from equation (3).

3.1. One-soliton equations

We consider an example of nonlinear PDEs which admit at least one-parametric solution. Let $N = 1$ in equation (2), $\boldsymbol{A} = a_1 \equiv a$. We will use the following solution of the characteristic equations (47), (48):

$$\omega_{11} = \frac{b\kappa_{32} - \kappa_{41}}{2\kappa_{32}} \qquad k_{11} = \frac{b\kappa_{32} - \kappa_{31}}{2\kappa_{32}} \tag{56}$$

where parameter $b \equiv b_1$ is introduced by equation (52). Then the compatibility condition (53) becomes a system of two equations and can be solved, for instance for parameters κ_{20} and γ_{02} (see appendix, equations (108), (109)). The system (32)–(40) admits the following solution:

$$\begin{aligned} &\gamma_{31} = -\gamma_{40} \qquad \gamma_{41} = 0 \\ &\alpha_1 = (\kappa_{41} - \kappa_{31})/(4\kappa_{32}) \\ &\alpha_2 = \tfrac{1}{2} \qquad \alpha_3 = \tfrac{-1}{2} \end{aligned} \tag{57}$$

and the system (41)–(44) can be solved, for instance for κ_{13}, κ_{12} and γ_{02}:

$$\begin{aligned} &\kappa_{13} = \kappa_{23} \qquad \kappa_{12} = (-2\kappa_{22}\kappa_{32} + 3\kappa_{23}(\kappa_{31} + \kappa_{41}))/(2\kappa_{32}) \\ &\gamma_{02} = -\gamma_{20} + \frac{\kappa_{23}(\kappa_{32} + 3\gamma_{40})}{2\kappa_{32}}. \end{aligned} \tag{58}$$

After substitution of the set of relations (108), (109), (57) and (58) into equations (97)–(107) one gets the following set of relations among the coefficients r_i of equation (45):

$$\begin{aligned} r_7 = {} & (27r_1r_3^3 - 27b^3r_3^3(6r_3 + r_4 + r_6) + r_2^2(r_2r_4 - 3r_3r_5 + r_2r_6) \\ & -9br_2r_3(-2r_3r_5 + r_2(2r_3 + r_4 + r_6)) \\ & +27b^2r_3^2(-(r_3r_5) + r_2(4r_3 + r_4 + r_6)))/(9r_3^2(-r_2 + 3br_3)) \end{aligned} \tag{59}$$

$$r_8 = -12r_3 - 2r_4 - 3r_6 \tag{60}$$

$$r_9 = -2r_2 - r_5 \tag{61}$$

$$r_{10} = 6r_3 + r_4 + r_6 \tag{62}$$

where r_i $(i = 1, \ldots, 6)$ are arbitrary parameters.

Now we construct the solution of equation (45), related with the scalar algebraic equation (2). One gets from equations (46) and (55)

$$\boldsymbol{f} = f_1 \mathrm{e}^{\omega_{11}x+\omega_{12}t} \qquad \boldsymbol{g} = g_1 \mathrm{e}^{k_{11}x+k_{12}t} \qquad \boldsymbol{R} = \frac{f_1 g_1}{\omega_{11} + k_{11}} \mathrm{e}^{(\omega_{11}+k_{11})x+(\omega_{12}+k_{12})t} \tag{63}$$

where parameters ω_{1j} and k_{1j} $(j = 1, 2)$ are related with parameters κ_{ij}, γ_{ij} and b by the formulae (49), (50) and (56), and the function u (54) is the travelling wave solution which can be written in the form

$$u = \frac{A}{\alpha} \tanh(\alpha(x - vt + \phi)) \tag{64}$$

or

$$u_x = \frac{A}{\cosh^2(\alpha(x - vt + \phi))} \tag{65}$$

$$\alpha = \tfrac{1}{2}(\omega_{11} + k_{11}) \tag{66}$$

$$A = -\alpha^2 = -\tfrac{1}{4}(\omega_{11} + k_{11})^2 \qquad v = -\frac{\omega_{12} + k_{12}}{\omega_{11} + k_{11}} \tag{67}$$

$$\phi = \frac{\ln(-f_1 g_1/(2\alpha))}{2\alpha} \qquad \mathrm{sign}(\alpha) \neq \mathrm{sign}(f_{10} g_{10}). \tag{68}$$

One can see that equation (64) represents a kink, and its x-derivative, given by equation (65), is a soliton. For the analysis we will use the x-derivative of the function u rather than function u itself. In accordance with this we call A and v amplitude and velocity of the travelling wave solution respectively. Relations (108), (109) and (57), (58) reduce formulae (66) and (67) to the following ones:

$$\alpha = \frac{3r_3b - r_2}{6r_3} \tag{69}$$

$$A = -\frac{(3r_3b - r_2)^2}{36r_3^2} \tag{70}$$

$$v = \frac{2r_2^3 - 9br_2^2r_3 + 27r_3^2(b^3r_3 + r_1)}{9r_3(3br_3 - r_2)}. \tag{71}$$

Let us use the square root of the negative amplitude as independent parameter $B = \sqrt{-A}$ and express the velocity in terms of this parameter. For this purpose one needs to solve equation (70) for b and substitute it into equation (71). One has

$$b_\pm = \frac{1}{3}\left(\frac{r_2}{r_3} \pm 6B\right) \qquad \alpha = B \tag{72}$$

$$v_\pm = 4B^2r_3 \pm \left(\frac{r_1}{2B} + 2Br_2\right) \qquad A = -B^2. \tag{73}$$

Formulae (59) and (64) show that the parameter b appears in both the equation and its solution. This means that equation (45) with relation among coefficients r_i given by the system (59)–(62) admits only one solution constructed by the algorithm developed in this section. In the particular case if

$$r_1 = 0 \qquad r_5 = -2r_2 \qquad r_6 = -6r_3 - r_4 \tag{74}$$

parameter b disappears from equation (45) so that it has one-parametric solution (64). This equation has the following form:

$$u_t + r_2(u_{xx} - 2uu_x) + r_3u_{xxx} + r_4u_x^2 - (r_4 + 6r_3)(uu_{xx} - u^2u_x) = 0. \tag{75}$$

This becomes an integrated KdV equation if $r_2 = 0$ and $r_4 + 6r_3 = 0$. If $r_4 = -3r_3$, equation (75) becomes a differentiated Burgers-type equation which can be linearized by the Hopf substitution $u = -(\ln f)_x$ [20, 21].

3.2. *Two-soliton equations*

In this section we consider an example of PDEs which admit both one- and two-parametric solutions. Let $N = 2$ in equation (2) and $\boldsymbol{A} = \mathrm{diag}(a_1, a_2)$. We will use the following solution of the characteristic equations (47), (48):

$$\omega_{i1} = \frac{b_i\kappa_{32} - \kappa_{41}}{2\kappa_{32}} \qquad k_{i1} = \frac{b_i\kappa_{32} - \kappa_{31}}{2\kappa_{32}} \qquad i = 1, 2 \tag{76}$$

where parameters b_1 and b_2 have been introduced by equation (52). Then the system (53), which represents compatibility condition, can be solved, for instance for the parameters $\kappa_{20}, \kappa_{21}, \kappa_{11}, \gamma_{01}, \gamma_{30}, \gamma_{31}, \gamma_{40}, \gamma_{41}$ (see appendix equations (110)–(117)). With the found expressions for γ_{31}, γ_{40} and γ_{41} the system (32)–(40) has the following solution:

$$\alpha_1 = (-\kappa_{31} + \kappa_{41})/(4\kappa_{32}) \qquad \alpha_2 = \tfrac{1}{2} \qquad \alpha_3 = \tfrac{-1}{2} \tag{77}$$

and system (41)–(44) can be solved for κ_{13}, κ_{12} and γ_{02}

$$\begin{aligned} &\kappa_{13} = \kappa_{23} \qquad \kappa_{12} = (-2\kappa_{22}\kappa_{32} + 3\kappa_{23}(\kappa_{31} + \kappa_{41}))/(2\kappa_{32}) \\ &\gamma_{02} = -\gamma_{20} + \frac{\kappa_{23}(\kappa_{31} - 4b_1\kappa_{32} - 4b_2\kappa_{32} + \kappa_{41})}{2(\kappa_{31} - b_1\kappa_{32} - b_2\kappa_{32} + \kappa_{41})}. \end{aligned} \tag{78}$$

Substituting expressions (110)–(117), (77) and (78) into equations (97)–(107) one gets the following set of relations among the coefficients r_i of equation (45):

$$r_4 = (3r_3(-2r_2^3 + 12(b_1 + b_2)r_2^2 r_3 - 18(b_1^2 + 3b_1 b_2 + b_2^2)r_2 r_3^2 \\ +9r_3^2(6r_3 b_1 b_2(b_1 + b_2) - r_1)))/(((r_2 - 3b_1 r_3)(r_2 - 3b_2 r_3) \\ \times(2r_2 - 3(b_1 + b_2)r_3)) \quad (79)$$

$$r_5 = \frac{-2(r_2^3 - 3(b_1 + b_2)r_2^2 r_3 + 9r_1 r_3^2 + 9b_1 b_2 r_2 r_3^2)}{(r_2 - 3b_1 r_3)(r_2 - 3b_2 r_3)} \quad (80)$$

$$r_6 = \frac{6r_2 r_3}{-2r_2 + 3(b_1 + b_2)r_3} \quad (81)$$

$$r_7 = \frac{3r_1 r_3(5r_2^2 - 15(b_1 + b_2)r_2 r_3 + 9(b_1^2 + 3b_1 b_2 + b_2^2)r_3^2)}{(r_2 - 3b_1 r_3)(r_2 - 3b_2 r_3)(-2r_2 + 3(b_1 + b_2)r_3)} \quad (82)$$

$$r_8 = \frac{6r_3(r_2^3 - 3(b_1 + b_2)r_2^2 r_3 + 9r_1 r_3^2 + 9b_1 b_2 r_2 r_3^2)}{(r_2 - 3b_1 r_3)(r_2 - 3b_2 r_3)(2r_2 - 3(b_1 + b_2)r_3)} \quad (83)$$

$$r_9 = \frac{18r_1 r_3^2}{(r_2 - 3b_1 r_3)(r_2 - 3b_2 r_3)} \quad (84)$$

$$r_{10} = \frac{27r_1 r_3^3}{(r_2 - 3b_1 r_3)(r_2 - 3b_2 r_3)(-2r_2 + 3(b_1 + b_2)r_3)} \quad (85)$$

where parameters r_1, r_2, r_3 are arbitrary.

Let us construct the solution of equations (45). The exact formula for this solution may be obtained from equation (54) by using the representation (46) for the functions $\boldsymbol{f}$ and $\boldsymbol{g}$ with $N = 2$ and equation (55):

$$u \equiv U_0 = -\partial_x \ln(\det(\boldsymbol{I} - \boldsymbol{R})) \quad (86)$$

$$\boldsymbol{f} = \begin{bmatrix} f_1 e^{k_{11}x + k_{12}t} \\ f_2 e^{k_{21}x + k_{22}t} \end{bmatrix} \qquad \boldsymbol{g} = [\, g_1 e^{\omega_{11}x + \omega_{12}t} \quad g_2 e^{\omega_{21}x + \omega_{22}t} \,] \qquad \boldsymbol{R} = \partial_x^{-1} \boldsymbol{f}\boldsymbol{g} \quad (87)$$

where parameters ω_{ij} and k_{ij} are related with parameters γ_{ij} and κ_{ij} by equations (49), (50) and (76). One gets the following expression for u:

$$u = -\partial_x \ln\left(1 - \frac{f_1 g_1}{k_{11} + \omega_{11}} e^{\eta_1} - \frac{f_2 g_2}{k_{21} + \omega_{21}} e^{\eta_2} + \varphi \frac{f_1 f_2 g_1 g_2}{(k_{11} + \omega_{11})(k_{21} + \omega_{21})} e^{\eta_1 + \eta_2}\right) \quad (88)$$

$$\eta_i = 2\alpha_i(x - v_i t)$$

$$\alpha_i = \tfrac{1}{2}(\omega_{i1} + k_{i1}) = \frac{3r_3 b_i - r_2}{6r_3} \quad (89)$$

$$v_i = -\frac{\omega_{i2} + k_{i2}}{\omega_{i1} + k_{i1}} = \frac{2r_2^3 - 9b_i r_2^2 r_3 + 27r_3^2(r_1 + b_i^3 r_3)}{9r_3(3b_i r_3 - r_2)} \qquad i = 1, 2 \quad (90)$$

$$\varphi = \frac{(k_{11} - k_{21})(\omega_{11} - \omega_{21})}{(k_{11} + \omega_{21})(k_{21} + \omega_{11})} = \frac{9(b_1 - b_2)^2 r_3^2}{(2r_2 - 3(b_1 + b_2)r_3)^2}. \quad (91)$$

Function u constructed in this way represents a two-kink solution of the nonlinear equation. The derivative, u_x, describes two solitons. In the case of elastic interaction ($\varphi \neq 0$) these solitons are far from each other as $t \to \pm\infty$. Their amplitudes and velocities are given by the formulae (70)

$$A_i = -\frac{(3r_3 b_i - r_2)^2}{36r_3^2} \qquad i = 1, 2 \quad (92)$$

and (90). The only result of the elastic interaction are phase shifts $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ of solitons which are equal to

$$\tilde{\varphi}_i = -\mathrm{sign}(\alpha_j(v_i - v_j))\frac{\ln(\varphi)}{2\alpha_i} \qquad i = 1, 2 \quad i \neq j \tag{93}$$

where α_i, v_i and φ are given by equations (89)–(91). One can see that parameters b_1 and b_2 appear in both coefficients of equation (45) and solution (88), which means that we can construct only one solution for equation (45) with relations among coefficients r_i given by the system (79)–(85). If $r_1 = r_2 = 0$, then parameters b_1 and b_2 disappear from equation (45) in which case it becomes integrated KdV:

$$u_t + r_3(u_{xxx} - 6u_x^2) = 0 \tag{94}$$

with the two-parametric solution given by equation (88). If $r_1 = r_3 = 0$, one gets the differentiated Burgers equation

$$u_t + r_2(u_{xx} - 2uu_x) = 0. \tag{95}$$

The solution (88) is a two-kink solution with two arbitrary parameters b_1 and b_2 and nonelastic interaction (after collision one has a single kink). If we require that only one parameter (say b_1) is arbitrary and $b_2 = -b_1$, $r_1 = 0$, then one gets

$$u_t + r_2(u_{xx} - 2uu_x) + r_3(u_{xxx} - 3u_x^2 - 3uu_{xx} + 3u^2u_x) = 0 \tag{96}$$

which is a differentiated Burgers-type equation. Formula (88) gives a two-kink solution, which interact inelastically. This solution is parametrized by the single parameter b_1. Recall that equation (96) can be linearized by the Hopf substitution $u = -(\ln f)_x$.

4. Conclusions

The dressing method discussed in this paper is a modification of the classical dressing method, based on the $\bar{\partial}$-problem [13–15]. Although the dressing method was originally developed for nonlinear PDEs integrable by the inverse scattering problem, this modification allows one to use the classical algorithm for construction of the particular solutions for nonintegrable systems of PDE. In the above example we have constructed the general system of nonlinear PDEs (15), (25) related with the given matrix equations (2)–(5), (18), (19) and introduced some reductions of this system with particular solutions. Of course, examples of one- and two-soliton (kink) equations, derived in sections 3.1 and 3.2 (equation (45) with (59)–(62) or (79)–(85) as well as equations (75) and (94)–(96)) do not cover all possible equations of this type.

Note that the manifold of available solutions is not as large as the manifold of available solutions in the classical dressing method. For instance, we have found only a *single* solution for equation (45) with (59)–(62) (as well as for equation (45) with (79)–(85)). This is because the type of particular solution affects the compatibility conditions (6), (20), which is one of the factors determining nonlinear PDEs. However, the importance of the dressing method lies not only in the construction of particular solutions, but also in revealing the link between the system of nonlinear PDEs (equations (15), (25) in our case) and the auxiliary overdetermined linear system of PDEs on the function ψ (equations (14), (24)), which has been derived in the intermediate step of the algorithm. The nonlinear system can be considered as the compatibility condition for the above linear system on the manifold of solutions, ψ, which is implicitly specified by equations (6) and (20). A further problem might be the construction of the operator representation for the compatibility condition of the overdetermined linear system (14), (24), which would give rise to the operator representation for the nonlinear system (15), (25) (similar to the zero-curvature representation for the classical integrable systems [11, 12]). However, importance of the operator representation in the nonintegrable case is not evident.

There are several other methods for study of nonlinear PDEs without relation to the complete integrability. The Hirota [22–25] and Painlevé [26–28] methods are well known in this area. The difference between these two methods and the method discussed in this paper is that both the Hirota and Painlevé methods deal with the nonlinear differential equation itself, while dressing methods arrive at the nonlinear PDE only after some special procedure. One has no criteria to check whether a given PDE can be treated by dressing method. However, dressing methods exhibit many specific properties of the *completely integrable equations*, which are difficult to determine by using Hirota and Painlevé methods. A similar situation exists in the nonintegrable case, when the represented modification of the dressing method works. But the properties of the nonintegrable PDE resulting from this method have not been presented yet. This problem is left for future studies.

Basic analytical calculations were done using Mathematica.

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Appendix

Expressions for the coefficients, r_i, of equation (45) in terms of parameters κ_{ij}, γ_{ij} and α_i:

$$r_1 = \frac{\alpha_1(2\kappa_{32}(\kappa_{21}-\kappa_{11}) + 3\kappa_{23}(\kappa_{30}-\kappa_{40})) - 2\kappa_{32}(\kappa_{10}+\kappa_{20})}{2\kappa_{32}} \tag{97}$$

$$r_2 = (-1+2\alpha_2)\kappa_{22} + (3\kappa_{23}(\alpha_1\kappa_{32}-\alpha_2\kappa_{41}))/(2\kappa_{32}) \tag{98}$$

$$r_3 = ((-2+3\alpha_2)\kappa_{23})/2 \tag{99}$$

$$\begin{aligned} r_4 = {} & -(-6(1+\alpha_3)\kappa_{23}\kappa_{32} + \alpha_2^2(3\gamma_{31}\kappa_{23} + (-2\gamma_{11}+\kappa_{23})\kappa_{32}) \\ & +\alpha_2(3\alpha_1\gamma_{41}\kappa_{23} + 2(\gamma_{11}+\kappa_{23})\kappa_{32}))/(2\kappa_{32}) \end{aligned} \tag{100}$$

$$\begin{aligned} r_5 = {} & -(3\alpha_1^2\gamma_{41}\kappa_{23} + 2\gamma_{10}\kappa_{32} - 2(3+4\alpha_3)\kappa_{22}\kappa_{32} \\ & +\alpha_1(3\gamma_{40}\kappa_{23} + 2\gamma_{11}\kappa_{32} - 4\gamma_{20}\kappa_{32} + 9\kappa_{23}\kappa_{32}) + 6\alpha_3\kappa_{23}\kappa_{41} \\ & +\alpha_2(3\gamma_{30}\kappa_{23} + 6\alpha_1\gamma_{31}\kappa_{23} + 2\gamma_{01}\kappa_{32} - 2\gamma_{10}\kappa_{32} - 4\alpha_1\gamma_{11}\kappa_{32} \\ & +4\kappa_{22}\kappa_{32} + 2\alpha_1\kappa_{23}\kappa_{32} - 3\kappa_{23}\kappa_{41}))/(2\kappa_{32}) \end{aligned} \tag{101}$$

$$r_6 = -(2(\gamma_{20} - (4+3\alpha_3)\kappa_{23})\kappa_{32} + \alpha_2(3\gamma_{40}\kappa_{23} + 3\alpha_1\gamma_{41}\kappa_{23} - 4\gamma_{20}\kappa_{32} + 7\kappa_{23}\kappa_{32}))/(2\kappa_{32}) \tag{102}$$

$$\begin{aligned} r_7 = {} & (2(-\gamma_{00}+\kappa_{21})\kappa_{32} - \alpha_{11}^2(3\gamma_{31}\kappa_{23} + (-2\gamma_{11}+\kappa_{23})\kappa_{32}) \\ & +\alpha_3(2(-\kappa_{11}+\kappa_{21})\kappa_{32} + 3\kappa_{23}(\kappa_{30}-\kappa_{40})) \\ & +\alpha_1(-3\gamma_{30}\kappa_{23} - 2\gamma_{01}\kappa_{32} + 2\gamma_{10}\kappa_{32} - 4\kappa_{22}\kappa_{32} + 3\kappa_{23}\kappa_{41}))/(2\kappa_{32}) \end{aligned} \tag{103}$$

$$\begin{aligned} r_8 = {} & (6(\gamma_{20} - 2\kappa_{23})\kappa_{32} - \alpha_3(6\gamma_{40}\kappa_{23} + 9\alpha_1\gamma_{41}\kappa_{23} + 2\gamma_{11}\kappa_{32} - 8\gamma_{20}\kappa_{32} + 16\kappa_{23}\kappa_{32}) \\ & +\alpha_2(3\gamma_{40}\kappa_{23} + 6\alpha_1\gamma_{41}\kappa_{23} + 2\gamma_{11}\kappa_{32} - 4\gamma_{20}\kappa_{32} + 3\kappa_{23}\kappa_{32} \\ & -2\alpha_3(3\gamma_{31}\kappa_{23} - 2\gamma_{11}\kappa_{32} + \kappa_{23}\kappa_{32})))/(2\kappa_{32}) \end{aligned} \tag{104}$$

$$\begin{aligned} r_9 = {} & (3\alpha_1^2\gamma_{41}\kappa_{23} + 2(\gamma_{10} - \kappa_{22})\kappa_{32} + \alpha_1(3\gamma_{40}\kappa_{23} + 2\gamma_{11}\kappa_{32} - 4\gamma_{20}\kappa_{32} + 3\kappa_{23}\kappa_{32}) \\ & -\alpha_3(3\gamma_{30}\kappa_{23} + 6\alpha_1\gamma_{31}\kappa_{23} + 2\gamma_{01}\kappa_{32} - 2\gamma_{10}\kappa_{32} - 4\alpha_1\gamma_{11}\kappa_{32} \\ & +4\kappa_{22}\kappa_{32} + 2\alpha_1\kappa_{23}\kappa_{32} - 3\kappa_{23}\kappa_{41}))/(2\kappa_{32}) \end{aligned} \tag{105}$$

$$r_{10} = (2(-\gamma_{20} + \kappa_{23})\kappa_{32} + \alpha_3(3\gamma_{40}\kappa_{23} + 6\alpha_1\gamma_{41}\kappa_{23} + 2\gamma_{11}\kappa_{32} - 4\gamma_{20}\kappa_{32} + 3\kappa_{23}\kappa_{32}) - \alpha_3^2(3\gamma_{31}\kappa_{23} + (-2\gamma_{11} + \kappa_{23})\kappa_{32}))/(2\kappa_{32}) \quad (106)$$

$$r_{11} = (-2\kappa_{21}\kappa_{32} + \alpha_2(2(-\kappa_{11} + \kappa_{21})\kappa_{32} + 3\kappa_{23}(\kappa_{30} - \kappa_{40})) + \alpha_{11}(4\kappa_{22}\kappa_{32} - 3\kappa_{23}\kappa_{41}))/(2\kappa_{32}). \quad (107)$$

Particular solution of system (53) with $N = 1$:

$$\kappa_{20} = (b^3(2\gamma_{02} + 2\gamma_{11} + 2\gamma_{20} - \kappa_{13} - \kappa_{23})\kappa_{32}^3 - \gamma_{11}\kappa_{31}^2\kappa_{41} + 2\gamma_{01}\kappa_{31}\kappa_{32}\kappa_{41} + 2\gamma_{10}\kappa_{31}\kappa_{32}\kappa_{41} - 4\gamma_{00}\kappa_{32}^2\kappa_{41} + 4\kappa_{21}\kappa_{32}^2\kappa_{41} - \gamma_{11}\kappa_{31}\kappa_{41}^2 - \gamma_{20}\kappa_{31}\kappa_{41}^2 + 2\gamma_{10}\kappa_{32}\kappa_{41}^2 - 2\kappa_{22}\kappa_{32}\kappa_{41}^2 - \gamma_{20}\kappa_{41}^3 + \kappa_{23}\kappa_{41}^3 - \gamma_{02}\kappa_{31}^2(\kappa_{31} + \kappa_{41}) + b^2\kappa_{32}^2(-(\gamma_{20}\kappa_{31}) + 3\kappa_{13}\kappa_{31} + 4\gamma_{01}\kappa_{32} + 4\gamma_{10}\kappa_{32} - 2\kappa_{12}\kappa_{32} - 2\kappa_{22}\kappa_{32} - 5\gamma_{20}\kappa_{41} + 3\kappa_{23}\kappa_{41} - 3\gamma_{11}(\kappa_{31} + \kappa_{41}) - \gamma_{02}(5\kappa_{31} + \kappa_{41})) + b\kappa_{32}(-3\kappa_{13}\kappa_{31}^2 - 6\gamma_{01}\kappa_{31}\kappa_{32} - 2\gamma_{10}\kappa_{31}\kappa_{32} + 4\kappa_{12}\kappa_{31}\kappa_{32} + 8\gamma_{00}\kappa_{32}^2 - 4\kappa_{11}\kappa_{32}^2 - 4\kappa_{21}\kappa_{32}^2 + 2\gamma_{20}\kappa_{31}\kappa_{41} - 2\gamma_{01}\kappa_{32}\kappa_{41} - 6\gamma_{10}\kappa_{32}\kappa_{41} + 4\kappa_{22}\kappa_{32}\kappa_{41} + 4\gamma_{20}\kappa_{41}^2 - 3\kappa_{23}\kappa_{41}^2 + 2\gamma_{02}\kappa_{31}(2\kappa_{31} + \kappa_{41}) + \gamma_{11}(\kappa_{31}^2 + 4\kappa_{31}\kappa_{41} + \kappa_{41}^2)) + \kappa_{13}\kappa_{31}^3 + 2\gamma_{01}\kappa_{31}^2\kappa_{32} - 2\kappa_{12}\kappa_{31}^2\kappa_{32} - 4\gamma_{00}\kappa_{31}\kappa_{32}^2 + 4\kappa_{11}\kappa_{31}\kappa_{32}^2 - 8\kappa_{10}\kappa_{32}^3)/(8\kappa_{32}^3) \quad (108)$$

$$\gamma_{30} = (2\gamma_{31}\kappa_{31}\kappa_{32} - b^2\gamma_{41}\kappa_{32}^2 + (-(\gamma_{41}\kappa_{31}) + 2\gamma_{40}\kappa_{32})\kappa_{41} + b\kappa_{32}(-2(\gamma_{31} + \gamma_{40})\kappa_{32} + \gamma_{41}(\kappa_{31} + \kappa_{41}))))/(4\kappa_{32}^2). \quad (109)$$

Particular solution of system (53) with $N = 2$:

$$\kappa_{20} = (b_1^2 b_2(2\gamma_{11} + \kappa_{13} + \kappa_{23})\kappa_{32}^3 + b_1 b_2^2(2\gamma_{11} + \kappa_{13} + \kappa_{23})\kappa_{32}^3 - b_1^2\kappa_{32}^2(2\gamma_{11}\kappa_{31} + \kappa_{13}\kappa_{31} + \gamma_{20}(\kappa_{31} - \kappa_{41}) + \kappa_{23}\kappa_{41}) - b_2^2\kappa_{32}^2(2\gamma_{11}\kappa_{31} + \kappa_{13}\kappa_{31} + \gamma_{20}(\kappa_{31} - \kappa_{41}) + \kappa_{23}\kappa_{41}) - 2(\kappa_{13}\kappa_{31}^3 - \gamma_{10}\kappa_{31}^2\kappa_{32} - \kappa_{12}\kappa_{31}^2\kappa_{32} + 4\kappa_{10}\kappa_{32}^3 + \gamma_{20}\kappa_{31}^2\kappa_{41} + \gamma_{10}\kappa_{32}\kappa_{41}^2 - \kappa_{22}\kappa_{32}\kappa_{41}^2 - \gamma_{20}\kappa_{41}^3 + \kappa_{23}\kappa_{41}^3 + \gamma_{11}\kappa_{31}^2(\kappa_{31} + \kappa_{41})) - 2b_1 b_2\kappa_{32}^2(2\kappa_{13}\kappa_{31} - \kappa_{12}\kappa_{32} - \kappa_{22}\kappa_{32} + \gamma_{20}(\kappa_{31} - \kappa_{41}) + 2\kappa_{23}\kappa_{41} + \gamma_{11}(3\kappa_{31} + \kappa_{41})) + b_1\kappa_{32}(3\kappa_{13}\kappa_{31}^2 - 2\gamma_{10}\kappa_{31}\kappa_{32} - 2\kappa_{12}\kappa_{31}\kappa_{32} + 2\gamma_{10}\kappa_{32}\kappa_{41} - 2\kappa_{22}\kappa_{32}\kappa_{41} + 3\kappa_{23}\kappa_{41}^2 + 2\gamma_{11}\kappa_{31}(2\kappa_{31} + \kappa_{41}) + \gamma_{20}(\kappa_{31}^2 + 2\kappa_{31}\kappa_{41} - 3\kappa_{41}^2)) + b_2\kappa_{32}(3\kappa_{13}\kappa_{31}^2 - 2\gamma_{10}\kappa_{31}\kappa_{32} - 2\kappa_{12}\kappa_{31}\kappa_{32} + 2\gamma_{10}\kappa_{32}\kappa_{41} - 2\kappa_{22}\kappa_{32}\kappa_{41} + 3\kappa_{23}\kappa_{41}^2 + 2\gamma_{11}\kappa_{31}(2\kappa_{31} + \kappa_{41}) + \gamma_{20}(\kappa_{31}^2 + 2\kappa_{31}\kappa_{41} - 3\kappa_{41}^2))))/(8\kappa_{32}^3) \quad (110)$$

$$\kappa_{21} = (-(\gamma_{02}\kappa_{31}^2) - \gamma_{11}\kappa_{31}^2 + 4\gamma_{00}\kappa_{32}^2 + b_1^2(\gamma_{20} - \kappa_{23})\kappa_{32}^2 + b_2^2(\gamma_{20} - \kappa_{23})\kappa_{32}^2 - b_1 b_2(\gamma_{02} + \gamma_{11} - \gamma_{20} + \kappa_{23})\kappa_{32}^2 - 4\gamma_{10}\kappa_{32}\kappa_{41} + 4\kappa_{22}\kappa_{32}\kappa_{41} + 3\gamma_{20}\kappa_{41}^2 - 3\kappa_{23}\kappa_{41}^2 + b_1\kappa_{32}(\gamma_{02}\kappa_{31} + \gamma_{11}\kappa_{31} + 2\gamma_{10}\kappa_{32} - 2\kappa_{22}\kappa_{32} - 3\gamma_{20}\kappa_{41} + 3\kappa_{23}\kappa_{41}) + b_2\kappa_{32}(\gamma_{02}\kappa_{31} + \gamma_{11}\kappa_{31} + 2\gamma_{10}\kappa_{32} - 2\kappa_{22}\kappa_{32} - 3\gamma_{20}\kappa_{41} + 3\kappa_{23}\kappa_{41}))/(4\kappa_{32}^2) \quad (111)$$

$$\kappa_{11} = (-(\gamma_{02}\kappa_{31}^2) - 3\kappa_{13}\kappa_{31}^2 + 4\gamma_{10}\kappa_{31}\kappa_{32} + 4\kappa_{12}\kappa_{31}\kappa_{32} + 4\gamma_{00}\kappa_{32}^2 - b_1^2(2\gamma_{11} + \gamma_{20} + \kappa_{13})\kappa_{32}^2 - b_2^2(2\gamma_{11} + \gamma_{20} + \kappa_{13})\kappa_{32}^2 - b_1 b_2(\gamma_{02} + 5\gamma_{11} + 3\gamma_{20} + \kappa_{13})\kappa_{32}^2 - 4\gamma_{20}\kappa_{31}\kappa_{41} - \gamma_{20}\kappa_{41}^2 - \gamma_{11}(2\kappa_{31} + \kappa_{41})^2 + b_1\kappa_{32}(\gamma_{02}\kappa_{31} + 2\gamma_{20}\kappa_{31} + 3\kappa_{13}\kappa_{31} - 2\gamma_{10}\kappa_{32} - 2\kappa_{12}\kappa_{32} + 3\gamma_{20}\kappa_{41} + 3\gamma_{11}(2\kappa_{31} + \kappa_{41})) + b_2\kappa_{32}(\gamma_{02}\kappa_{31} + 2\gamma_{20}\kappa_{31} + 3\kappa_{13}\kappa_{31} - 2\gamma_{10}\kappa_{32} - 2\kappa_{12}\kappa_{32} + 3\gamma_{20}\kappa_{41} + 3\gamma_{11}(2\kappa_{31} + \kappa_{41}))))/(4\kappa_{32}^2) \quad (112)$$

$$\gamma_{01} = (-(b_1(\gamma_{02} + 2\gamma_{11} + \gamma_{20})\kappa_{32}) - b_2(\gamma_{02} + 2\gamma_{11} + \gamma_{20})\kappa_{32} + 2(\gamma_{02}\kappa_{31} - \gamma_{10}\kappa_{32} + \gamma_{20}\kappa_{41} + \gamma_{11}(\kappa_{31} + \kappa_{41}))))/(2\kappa_{32}) \quad (113)$$

$$\gamma_{30} = ((b_1 + b_2)\kappa_{32}(\kappa_{31} - \kappa_{41}))/(2(\kappa_{31} - b_1\kappa_{32} - b_2\kappa_{32} + \kappa_{41})) \tag{114}$$

$$\gamma_{31} = ((b_1 + b_2)\kappa_{32}^2)/(\kappa_{31} - b_1\kappa_{32} - b_2\kappa_{32} + \kappa_{41}) \tag{115}$$

$$\gamma_{40} = -\gamma_{31} \tag{116}$$

$$\gamma_{41} = 0. \tag{117}$$

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